

Normal ordering and boundary conditions in open bosonic strings

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Abstract

Boundary conditions play a non trivial role in string theory. For instance the rich structure of D-branes is generated by choosing appropriate combinations of Dirichlet and Neumann boundary conditions. Furthermore, when an antisymmetric background is present at the string end-points (corresponding to mixed boundary conditions) space time becomes non-commutative there.

We show here how to build up normal ordered products for bosonic string position operators that satisfy both equations of motion and open string boundary conditions at quantum level. We also calculate the equal time commutator of these normal ordered products in the presence of antisymmetric tensor background.

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1 Introduction

Recent progress in string theory[1] indicates a scenario where our four dimensional space-time should correspond to a D-brane[2] representing the boundary of a larger manifold. This idea also proved useful indicating a possible explanation for the hierarchy problem[3, 4]. One important consequence of such a model is the non-commutativity of space time coordinates in our four dimensional world[5, 6, 7]. The reason is that D-branes correspond to the space where open string endpoints are located and where the corresponding string boundary conditions must be satisfied. In the presence of an antisymmetric tensor background these conditions are incompatible with commuting coordinates.

Since antisymmetric fields show up in the massless spectrum of closed strings living on the D-branes it is reasonable to suspect that our physical world could be non-commutative at very small length scales. This is one of the reasons for the increasing interest in studying many aspects of non-commutative quantum field theories as can be seen for example in[7, 8]. Furthermore, this fact illustrates the non trivial role of boundary conditions in string theory and the importance of taking them into account when considering the quantization of open strings.

In quantum field theory, products of quantum fields at the same space-time points are in general singular objects. The same thing happens in string theory if one multiplies position operators, that can be taken as conformal fields on the world sheet. This situation is well known and one can remove the singular part of the operator products by defining normal ordered well behaved objects[9]. This is important, for example, when one builds up the generators of conformal transformations and investigate the realization, at quantum level, of the classical symmetries.

Normal ordered products of operators are usually defined so as to satisfy the classical equations of motion at quantum level. The purpose of this article is to define normal ordered products for open string position operators that additionally satisfy the boundary conditions. This way we will define a normal ordering that will be valid also at string end-points. We will also investigate the relation between this new definition for normal ordering and the non commutativity of space time coordinates.

2 String position operator products

The classical action for a bosonic string in the presence of a constant antisymmetric background taking a world sheet with Euclidean signature is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \left(g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - i \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \quad (1)$$

where X^μ are the spacetime string coordinates and $B_{\mu\nu}$ is the antisymmetric field. The string world sheet Σ is represented by the parameters $\sigma_1 \equiv \tau$, $\sigma_2 \equiv \sigma$ with, as usual, the boundary (string endpoints) at $\sigma = 0, \pi$. The Euclidean world sheet metric is $g^{\tau\tau} = g^{\sigma\sigma} = 1$ and the antisymmetric tensor is chosen by $\epsilon^{\tau\sigma} = 1$.

The variation of the action gives us a volume term that vanishes imposing the equations of motion

$$(\partial_\tau^2 + \partial_\sigma^2)X^\mu = 0 \quad (2)$$

plus a boundary term that vanishes if we additionally impose that the string coordinates satisfy the boundary conditions

$$\begin{aligned} (\eta_{\mu\nu}\partial_\sigma X^\nu + iB_{\mu\nu}\partial_\tau X^\nu)|_{\sigma=0} &= 0 \\ (\eta_{\mu\nu}\partial_\sigma X^\nu + iB_{\mu\nu}\partial_\tau X^\nu)|_{\sigma=\pi} &= 0 \end{aligned} \quad (3)$$

This boundary conditions, when imposed at quantum level, are responsible for the non commutativity of the position operators[5, 7]. We can infer this result by realizing that this conditions represent constraints in phase space relating position and conjugate momenta.

It is convenient, for studying the quantum operators, to introduce complex world sheet coordinates: $z = \tau + i\sigma$, $\bar{z} = \tau - i\sigma$; $\partial_z = 1/2(\partial_\tau - i\partial_\sigma)$, $\partial_{\bar{z}} = 1/2(\partial_\tau + i\partial_\sigma)$

The action takes the form

$$S = \frac{1}{2\pi\alpha} \int dz^2 [\eta_{\mu\nu}\partial_z X^\mu \partial_{\bar{z}} X^\nu - B_{\mu\nu}\partial_z X^\mu \partial_{\bar{z}} X^\nu] \quad (4)$$

while classical equations of motion and boundary conditions take the form

$$\partial_{\bar{z}}\partial_z X^i = 0 \quad (5)$$

$$\begin{aligned} (\eta_{\mu\nu}(\partial_z - \partial_{\bar{z}}) + B_{\mu\nu}(\partial_z + \partial_{\bar{z}}))X^\nu|_{z=\bar{z}} &= 0 \\ (\eta_{\mu\nu}(\partial_z - \partial_{\bar{z}}) + B_{\mu\nu}(\partial_z + \partial_{\bar{z}}))X^\nu|_{z=\bar{z}+2\pi i} &= 0 \end{aligned} \quad (6)$$

We can study the properties of quantum operators by considering the expectation values of the corresponding classical objects. Defining the expectation value of an operators \mathcal{F} as[9]

$$\langle \mathcal{F}[X] \rangle = \int [dX] \exp(-S[X]) \mathcal{F}[X] \quad (7)$$

and using the fact that the path integral of a total derivative vanishes one finds that the equations of motion and boundary conditions are realized for the expectation values of string coordinates X^ν

$$\begin{aligned} 0 &= \int [dX] \frac{\delta}{\delta X^\nu(z', \bar{z}')} \exp(-S[X]) = \left\langle \frac{1}{\pi\alpha'} \partial_{\bar{z}'} \partial_{z'} X_\nu(z', \bar{z}') \right\rangle \\ &+ \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} \delta^2(z - z') \left\langle (\eta_{\nu\mu}(\partial_z - \partial_{\bar{z}}) + B_{\nu\mu}(\partial_z + \partial_{\bar{z}})) X^\mu(z, \bar{z}) dz \right\rangle = 0. \end{aligned} \quad (8)$$

The last (singular) term is integrated over the boundary, where $dz = d\bar{z}$. This equation implies that both string equations of motion and boundary condition hold as expectation values. So the corresponding quantum position operators satisfy the

equivalent conditions (as long as they are not multiplied by other operators located at the same world sheet point)

$$\partial_{\bar{z}}\partial_z\hat{X}^\nu(z,\bar{z}) = 0 \quad (9)$$

$$\begin{aligned} \left(\eta_{\nu\mu}(\partial_z - \partial_{\bar{z}}) + B_{\nu\mu}(\partial_z + \partial_{\bar{z}})\right)\hat{X}^\mu|_{z=\bar{z}} &= 0 \\ \left(\eta_{\nu\mu}(\partial_z - \partial_{\bar{z}}) + B_{\nu\mu}(\partial_z + \partial_{\bar{z}})\right)\hat{X}^\mu|_{z=\bar{z}+2\pi i} &= 0 \end{aligned} \quad (10)$$

Products of operators at the same point will have a singular behavior. We can see this by calculating

$$\begin{aligned} 0 &= \int [dX] \frac{\delta}{\delta X^\nu(z', \bar{z}')} \exp(-S[X]) X^\rho(z'', \bar{z}'') \\ &= \left\langle \delta^2(z' - z'') \delta_{\nu}^{\rho} + \left(\frac{1}{\pi\alpha'} \partial_{\bar{z}'} \partial_{z'} X_\nu(z', \bar{z}') X^\rho(z'', \bar{z}'') \right) \right. \\ &\quad \left. + \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} \delta^2(z - z') \left(\eta_{\nu\mu}(\partial_z - \partial_{\bar{z}}) + B_{\nu\mu}(\partial_z + \partial_{\bar{z}})\right) X^\mu(z, \bar{z}) X^\rho(z'', \bar{z}'') dz \right\rangle = 0. \end{aligned} \quad (11)$$

The volume term gives an extra singular term to the equation of motion for a product of two fields

$$\frac{1}{\pi\alpha'} \left\langle \partial_{z'} \partial_{\bar{z}'} X^\mu(z', \bar{z}') X^\nu(z'', \bar{z}'') \right\rangle = -\eta^{\mu\nu} \left\langle \delta^2(z' - z'', \bar{z}' - \bar{z}'') \right\rangle \quad (12)$$

while the boundary terms vanishes if this product of two fields satisfies the same boundary condition as the single field

$$\left\langle \left(\eta_{\nu\mu}(\partial_{z'} - \partial_{\bar{z}'}) + B_{\nu\mu}(\partial_{z'} + \partial_{\bar{z}'})\right) X^\mu(z', \bar{z}') X^\rho(z'', \bar{z}'')|_{Bound.} \right\rangle = 0, \quad (13)$$

where *Bound.* means that we are taking this condition both at $z = \bar{z}$ and at $z = \bar{z} + 2\pi i$. Thus the products of operators will satisfy

$$\partial_{\bar{z}'} \partial_{z'} \hat{X}^\mu(z', \bar{z}') \hat{X}^\nu(z'', \bar{z}'') = -\pi\alpha' \eta^{\mu\nu} \delta^2(z' - z'', \bar{z}' - \bar{z}'') \quad (14)$$

$$\left(\eta_{\nu\mu}(\partial_{z'} - \partial_{\bar{z}'}) + B_{\nu\mu}(\partial_{z'} + \partial_{\bar{z}'})\right) \hat{X}^\mu(z', \bar{z}') \hat{X}^\rho(z'', \bar{z}'')|_{Bound.} = 0 \quad (15)$$

If we define a normal ordered product of two position operators in the standard way [9]

$$: \hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') : = \hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') + \frac{\alpha'}{2} \eta^{\mu\nu} \ln|z - z'|^2 \quad (16)$$

it satisfies the equation of motion at quantum level:

$$\partial_{\bar{z}} \partial_z : \hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') : = 0 \quad (17)$$

but fails to satisfy the boundary conditions. So we will introduce a different kind of normal ordered product satisfying both equation of motion and boundary conditions.

The mathematical problem posed by defining the normal ordering is related to that of calculating Green's functions[10, 11, 12, 13]. The normal ordered product is defined by subtracting out the corresponding Green's functions. So we can find normal ordered products satisfying open string boundary condition using the solutions to open string Green's functions.

At this point it is more convenient to choose world sheet coordinates that simplify the representation of the boundary. In the present coordinates the boundary $\sigma = 0$ corresponds to $z = \bar{z}$, and $\sigma = \pi$ to $z = \bar{z} + 2\pi i$. Introducing

$$w = e^{\tau+i\sigma} \ ; \ \bar{w} = e^{\tau-i\sigma}$$

the complete boundary corresponds just to the region $w = \bar{w}$. On the other hand the factor $w\bar{w}$ in $\partial_z \partial_{\bar{z}} = w\bar{w} \partial_w \partial_{\bar{w}}$ cancel out precisely the Jacobian of the coordinate transformation in such a way that the action in terms of w, \bar{w} has still the same form as in eq. (4). The boundary conditions take the form

$$\begin{aligned} & \left(\eta_{\mu\nu} (w \partial_w - \bar{w} \partial_{\bar{w}}) + B_{\mu\nu} (w \partial_w + \bar{w} \partial_{\bar{w}}) \right) \hat{X}^\nu|_{w=\bar{w}} \\ &= w \left(\eta_{\mu\nu} (\partial_w - \partial_{\bar{w}}) + B_{\mu\nu} (\partial_w + \partial_{\bar{w}}) \right) \hat{X}^\nu|_{w=\bar{w}} = 0. \end{aligned} \quad (18)$$

This implies that starting with a solution in coordinates z, \bar{z} that satisfies the boundary conditions just at $\sigma = 0$ and replacing everywhere z, \bar{z} by w, \bar{w} we get a new solution that satisfies the boundary conditions both at $\sigma = 0$ and $\sigma = \pi$.

So our new normal ordering is defined as

$$\begin{aligned} : \hat{X}^\mu(w, \bar{w}) \hat{X}^\nu(w', \bar{w}') : &= \hat{X}^\mu(w, \bar{w}) \hat{X}^\nu(w', \bar{w}') + \frac{\alpha'}{2} \eta^{\mu\nu} \ln|w - w'|^2 \\ &+ \frac{\alpha'}{2} \left([\eta + B]^{-1} [\eta - B] \right)^{\mu\nu} \ln(w - \bar{w}') \\ &+ \frac{\alpha'}{2} \left([\eta + B] [\eta - B]^{-1} \right)^{\mu\nu} \ln(\bar{w} - w') + \alpha' D^{\mu\nu} \end{aligned} \quad (19)$$

where $D^{\mu\nu}$ is a constant that may depend on B but not on the coordinates.

3 Equal time commutators

It is important to investigate the effect of this normal ordering on the commutators of position operators to check if the non commutativity of space time coordinates in the presence of the antisymmetric tensor background is changed. We can rewrite eq.(19) in a more convenient form for calculating the commutators:

$$\begin{aligned}
:\hat{X}^\mu(w, \bar{w}) \hat{X}^\nu(w', \bar{w}') : &= \hat{X}^\mu(w, \bar{w}) \hat{X}^\nu(w', \bar{w}') + \frac{\alpha'}{2} \eta^{\mu\nu} \ln|w - w'|^2 \\
&- \alpha' \eta^{\mu\nu} \ln|w - \bar{w}'| + \alpha' G^{\mu\nu} \ln|w - \bar{w}'|^2 \\
&+ \frac{1}{2\pi} \Theta^{\mu\nu} \ln\left(\frac{w - \bar{w}'}{\bar{w} - w'}\right) + \alpha' D^{\mu\nu}
\end{aligned} \tag{20}$$

where we introduced

$$\begin{aligned}
G^{\mu\nu} &= \left([\eta + B]^{-1} \eta [\eta - B]^{-1} \right)^{\mu\nu} \\
\Theta^{\mu\nu} &= -2\pi\alpha' \left([\eta + B]^{-1} B [\eta - B]^{-1} \right)^{\mu\nu}
\end{aligned} \tag{21}$$

Now we calculate the normal ordered commutator at boundary points $w = \bar{w} = \tau$, $w' = \bar{w}' = \tau'$ using the same choice for the constant $D^{\mu\nu}$ and the same procedure as in [7]

$$\begin{aligned}
:\hat{X}^\mu(\tau), \hat{X}^\nu(\tau') : &\equiv :\hat{X}^\mu(\tau) \hat{X}^\nu(\tau') : - :\hat{X}^\nu(\tau') \hat{X}^\mu(\tau) : \\
&= [\hat{X}^\mu(\tau), \hat{X}^\nu(\tau')] + \alpha' G^{\mu\nu} \ln((\tau - \tau')^2) - \frac{i}{2} \Theta^{\mu\nu} \epsilon(\tau - \tau') \\
&- \alpha' G^{\nu\mu} \ln((\tau' - \tau)^2) + \frac{i}{2} \Theta^{\nu\mu} \epsilon(\tau' - \tau) \\
&= [\hat{X}^\mu(\tau), \hat{X}^\nu(\tau')]
\end{aligned} \tag{22}$$

So the commutator does not get any extra contribution from the new normal ordering prescription. The equal time commutator thus keeps the same form calculated in [7] (see also [14, 15]).

$$:\hat{X}^\mu(\tau), \hat{X}^\nu(\tau) : = i\Theta^{\mu\nu} \tag{23}$$

Concluding, the new normal ordering for position operators that is consistent with both equations of motion and boundary conditions at quantum level does not spoil the previous results related to non commutativity of space time coordinates.

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